CRITERIUM FOR FUNCTION DEFININING OF FINAL TIME SHARING OF THE BASIC CLARK'S FLOW PRECEDENCE DIAGRAMMING (PDM) STRUCTURE

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Abstract: *In this paper the results of the theoretically-experimental researches of the criterion of quantification of the superponed (superposition of two or more values to create a new, resulting activity value) flow time with two or more local - autonomous flows in the network diagram of PDM (Precedence Diagramming Method) type on the basis of Clark's equations are presented. Computational solving of this basic variant of the general flow model through the network is being performed by the analytical and simulated procedures. The mathematical experiment has been realised by the program package Mathcad Professional.*

Keywords: *Superponed function, Clark's model, network plan.*

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1. INTRODUCTION

The basic model, for which Clark's equation of the equivalent flow has been defined, consists of the oriented graph, where two activities proceed parallely, have common beginning and flow to one terminal "event" (Figure 1.).

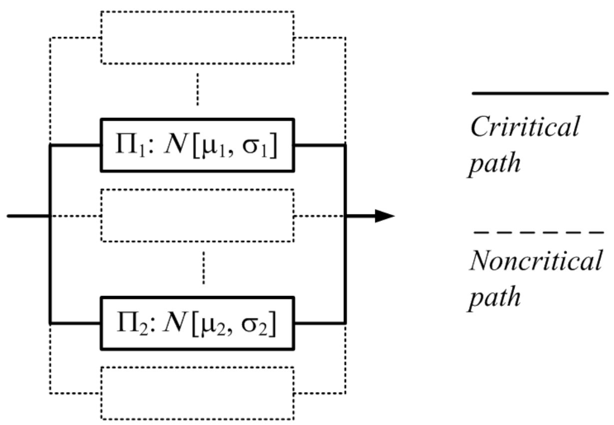

Figure 1. The flow network with the two locally - autonomous critical flows of PDM structure activity

In that sense, the activities can be locally – autonomous until they are completely realized. However, they can be interdependent, so Clark's equations were developed for that case, too [1]. In this work, the results of the basic Clark's equation are being compared with the results of numerical Monte-Carlo simulation. Such basic model with parallel flows of activities and events has a key role in the network planning. Both these methods, the analytical, as well as the numerical one, are characteristic for studying various appearances and processes based on the network models, whether they are the flows of activity, resources, energy, mass servicing, information, technical systems, reliability, emission of nuclear particles etc. Those problems are, as experience shows us [1], of stochastic nature, and it is often impossible to solve them in an analytical way (method), without certain approximation. This way, one network model of the flows of activities is defined and solved, stimulated by the researches of Clark, Styke [2], Dodin [3] and Haga and O'Keefe [4]. That also contributes to developing the algorithm for solving the general model of the critical flows established on the row-parallel structures of the oriented graph. In this analysis, the normal distribution of the endings of some activities with the characteristics of the average value and the adequate time deviation of their realization is superponed.

2. THE FLOWS WITH THE CRITICAL ACTIVITIES

The uniform solution of the critical activity flow, and also of the resulting flow time using the expected times of the separate activities, presents one of the most troublesome effects of the network application planning based on stochastic methods. The stochastic (but also the deterministic) networks of activities formed e.g. on the basis of AON (Activity On the Node - the method of the oriented graphs) structure, can be very complex in some cases of planning. The common issues of these flows are: the initial and the final event and the same, approximately the same or different values of the expected time of realization of critical that is, the sub critical flows. The final event will be realized if all the critical flows that "flow into it" have been realized. In that case, one can rightly ask: what is the certainty (as well as probability distribution) that the resulting flow time will be completed in the planned period, considering that such activity graph can contain one, two or an unlimited number of critical flows, primarily of the most complex, i.e. parallel type. To answer this question correctly, it is necessary to define exactly the criteria and the algorithm for the quantification of impacts, primarily of critical and sub critical flows on forming the resulting, i.e. the superponed flow time activities.

3. THE AIM OF THE PAPER

The basic aim of this work is the quantification of effects of the two critical flows on forming the resulting, i.e. the superponed flow time. The second objective is setting the criterion for defining the equivalence ($\equiv$) of the parallel flows. By solving it, we create the fundamental base for defining the function of probability distribution, as well as for the noticing of relativity of those flows by applying the Monte-Carlo method, as a control manner.

4. THE BASIC TIME PARAMETERS OF THE AUTONOMOUS CRITICAL FLOWS

According to the researches [1], [2], [4] the superposing of intervals of the critical and subcritical flow times and their dispersion and their reducing to one equivalent flow, can be calculated by:

- analytical methods: - Clark's equations for solving the parallel (as well as ordinal) flows,
- and by numerical method – Monte Carlo simulation for solving the parallel and the ordinal flows.

To illustrate the application of the listed basic algorithms, we can use the AON network with two parallel flows: Π_1 i Π_2 (Figure 1).

4.1 The superponed time and the flow variance

In structuring the algorithm for analytical solving of this option of critical flows, one starts from Clark's original equations. These equations solve the parameter flows as follows: the superponed flow time $T_{1,2}$ and its variances $\sigma^2(T_{1,2})$, in the condition of the non - existence of the correlation between the two activities. For the basic oriented graph with two parallel flows, from the initial (*i*) to the terminal (*j*) event (Figure 2) the average value of the flow time, $\overline{T_{1,2}}$ is:

- The superponed flow time for the independent parallel flows:

$$\overline{T_{1,2}} = \overline{T_1} \cdot \Phi(\xi_{1,2}) + \overline{T_2} \cdot \Phi(-\xi_{1,2}) + \lambda_{1,2} \cdot \Psi(\xi_{1,2}) \quad (1)$$

Where:

$\Phi(\xi) = \frac{1}{\sqrt{2\pi}} \cdot \int_0^{\xi} \exp(-\frac{1}{2} \cdot t^2) dt$ -is Laplace's integral,

$\Psi(\xi) = \frac{1}{\sqrt{2\pi}} \cdot \exp(-\frac{1}{2} \cdot t^2)$ -is the function of density of the centred normal distribution, and $\lambda_{1,2} = \sqrt{\sigma^2(T_1) + \sigma^2(T_2)}$, that is, $\xi_{1,2} = \frac{1}{\lambda_{1,2}} \cdot (\overline{T_1} - \overline{T_2})$ the parameters of Clark's functions.

In addition , we usually take the expected or the average values of time intervals: $\overline{T_1} = \mu_1$ and $\overline{T_2} = \mu_2$ and the standard deviations $\sigma_1 = \sigma(T_1)$ and $\sigma_2 = \sigma(T_2)$, so , according to [1], as follows:

- The average superponed flow time:

$$\mu_{1,2} = \mu_1 \cdot \Phi(\xi_{1,2}) + \mu_2 \cdot \Phi(-\xi_{1,2}) + \lambda_{1,2} \cdot \Psi(\xi_{1,2}) \quad (2)$$

- The superponed dispersion is presented by the next Clark's equation:

$$\sigma^2{}_{1,2} = \left(\mu_1^2 + \sigma^2{}_1)\right) \cdot \Phi(\xi_{1,2}) + \left(\mu_2^2 + \sigma^2{}_2)\right) \cdot \Phi(-\xi_{1,2}) + (\mu_1 + \mu_2) \cdot \lambda_{1,2} \cdot \varphi(\xi_{1,2}) - \mu_{1,2}^2 \quad (3)$$

By these equations we can describe the properties of one, equivalent flow instead of the previous two (Figure 2.).

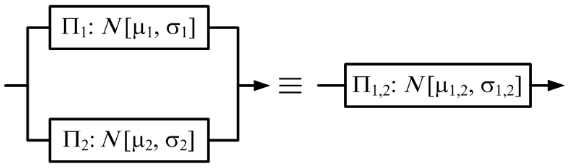

Figure 2. The superponed flow of activities

4.2 The growth of the superponed flow time in relation to the critical flow

On the basis of the new superponed function of time distribution $T_{1,2}$ with the characteristics $N[\mu_{1,2}, \sigma_{1,2}]$, the time growth $T_{1,2}$ can be quantified in relation to the single time T_1 or T_2 , depending on the fact which one of them has a critical feature. For the elementary network with autonomous flows Π_1 and Π_2, that growth or the *"superponed extract"*, after the simpler performing, is:

$$\Delta\mu_{1,2} = \lambda_{1,2} \cdot \varphi(\xi_{1,2}) + (\mu_2 - \mu_1) \cdot \Phi(-\xi_{1,2}) \qquad (4)$$

However, in the case of a reversed choice, it follows:

$$\Delta\mu_{2,1} = \lambda_{2,1} \cdot \varphi(\xi_{2,1}) + (\mu_1 - \mu_2) \cdot \Phi(-\xi_{2,1}) \qquad (5)$$

In addition to that, the nature of these values is always nonnegative, i.e.: $\Delta\mu_{1,2} \geq 0$ and $\Delta\mu_{2,1} \geq 0$.

4.3 The testing of the invariability of the flow model

The testing of invariability should show us if the derived values remained unchanged and uniformly fixed when the flow order in the calculating process was being changed. It is well known, that, when dealing with two flows with two parameters each, we can have nine relations for each flow. (Table 1). In other words, when analysing the following possible relations between the expected times and the deviations of single flows, as:

$$\mu_1 \begin{Bmatrix} < \\ = \\ > \end{Bmatrix} \mu_2 \text{ and } \sigma_1 \begin{Bmatrix} < \\ = \\ > \end{Bmatrix} \sigma_2, \text{ where the next}$$

relations are considered: $\rho = \begin{Bmatrix} < \\ = \\ > \end{Bmatrix}$. (6)

Table 1.Combination of relations of the two expected values and two standard deviations

$\mu_1 \rho \mu_2$	<	<	<	=	=	=	>	>	>
$\sigma_1 \rho \sigma_2$	<	=	>	<	=	>	<	=	>

We can conclude that nine different combinations can be formed here altogether.

$$\lambda_{1,2} = \lambda_{2,1}, \quad \xi_{1,2} = -\xi_{2,1},$$

$$\varphi(-\xi_{1,2}) = \varphi(\xi_{2,1}) \text{ and } \Phi(-\xi_{1,2}) = \Phi(\xi_{2,1}) \qquad (7)$$

Accepting that:

We get the invariant relations of the basic tested values which are related to the superponed flow, i.e.:

$$\mu_{1,2} = \mu_{2,1}; \ \Delta\mu_{1,2} = \Delta\mu_{2,1} \text{ and } \sigma^2(T_{1,2}) = \sigma^2(T_{2,1}) \qquad (8)$$

One can conclude that it is irrelevant which flow of the two observed will be declared as critical, and which one as subcritical. This invariability characteristic of models (4) and (5) is very important for developing the criterion of equivalence of *w*-flows [8] ($w \geq 2$).

5. THE APPLICATION OF THE SIMULATION MODELS

5.1 The application of Monte-Carlo method on models with parallel flows

Let's suppose that the elementary activities of the flow time have normal distribution with the parameters $N[\mu_\nu, \sigma_\nu], \ (\nu = 1, 2)$.

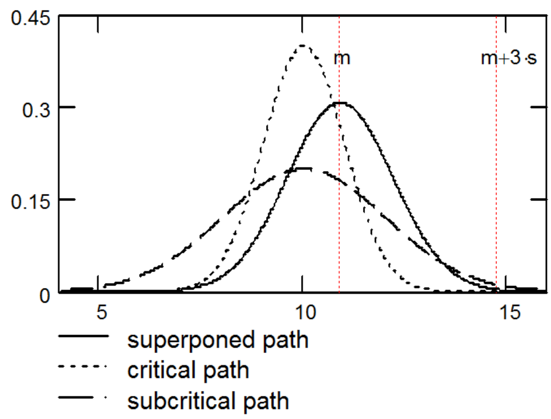

Figure 3. The distributions of probability of the critical, subcritical and superponed time flow

The results of the numeric simulation of $n = 10^5$ replications for the chosen characteristics: $N[\mu_1 = 10, \sigma_1 = 1]$, that is $N[\mu_2 = 10, \sigma_2 = 2]$ and the final result of simulation are presented in the form of the the simulated average value $m_{1,2}$ and the standard dispersion $s_{1,2}$, which generates the new distribution $N[m_{1,2}, s_{1,2}]$, (Figure 3.).

Here normal distributions are obtained with:

- Teoretical values by Clark's equations

$$N[\mu_{1,2} = 10{,}892062; \ \sigma_{1,2} = 1{,}305459],$$

- Simulation values:

$N\,[m_{1,2}=10{,}891189;\ s_{1,2}=1{,}301344]$ and

- Here the differences between the theoretical and the simulation values are:

$\Delta\mu_{1,2}=8{,}73\cdot10^{-4}$ and $\Delta\sigma_{1,2}=4{,}115\cdot10^{-3}$.

However, as this algorithm is simply defined by computer, the main point of the problem is now oriented to the domain of simulation. In other words, in one session of simulation of $n=10^5$ replications, the testing of only one chosen variant was performed here, where $\mu_1=\mu_2$ and $\sigma_1<\sigma_2$ of nine possible variants.

5.2 The application of Monte-Carlo method in solving the Clarks's flow model

The extension of Monte-Carlo method domain and the visualisation of its results can be done by using the frames [6]. In that sense the supposition (6) can be solved in up to the three variants in one simulation session:

$$\mu_1 \left\{\begin{matrix}<\\=\\<\end{matrix}\right\} \mu_2 \quad \text{and} \quad \sigma_1<\sigma_2 . \tag{9}$$

The number of frames depends on the complexity of the problem, i.e. the process which is being studied, so this integrated Monte-Carlo method - animation (through frames) has a significant role. It can be partially presented by a series of selected frames at work [8].

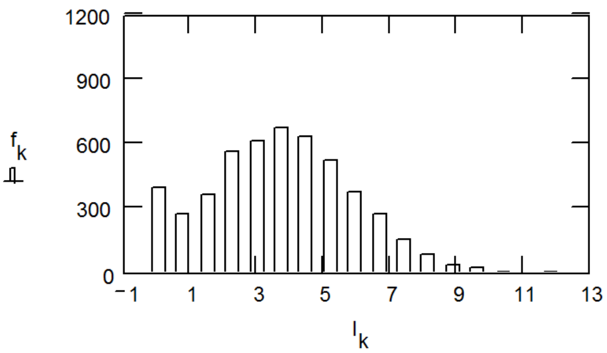

Figure 4. The value frame $[\mu_1=14,\ \sigma_1=1]$

5.3 Some criteria for determining the equivalence of flows

When the number of the parallel independent (absence of correlation) flows is $w\geq2$, then the criteria which are not completely universal can be developed, but they can be applied when the problem of superposition is solved analytically. Their definition can be applied in the next cases:

- The condition of the equivalence of the two parallel flows is expressed on the basis of two parameters and three set relations (Figure 1):

$$(\mu_1 \left\{\begin{matrix}>\\=\\<\end{matrix}\right\} \mu_2 \wedge \sigma_1=\sigma_2)\vee(\mu_1=\mu_2\wedge\sigma_1 \left\{\begin{matrix}>\\=\\<\end{matrix}\right\} \sigma_2) \tag{10}$$

$$\Rightarrow(\sigma_{1,2}\equiv\sigma_{2,1})\wedge(\mu_{1,2}\equiv\mu_{2,1})$$

This criterion is based on the evidence of the invariability of the two flows Π_1 and Π_2.

- Neither of $w=3$ parallel flows is equivalent in the next cases (Figure 4):

$$[(\mu_1\neq\mu_2\neq\mu_3)\wedge(\sigma_1 \left\{\begin{matrix}=\\ \neq\end{matrix}\right\} \sigma_2 \left\{\begin{matrix}=\\ \neq\end{matrix}\right\} \sigma_3)]\vee$$

$$[(\mu_1 \left\{\begin{matrix}=\\ \neq\end{matrix}\right\} \mu_2 \left\{\begin{matrix}=\\ \neq\end{matrix}\right\} \mu_3)\wedge(\sigma_1\neq\sigma_2\neq\sigma_3)]\Rightarrow \tag{11}$$

$$(\sigma_{1,2}\neq\sigma_{1,3}\neq\sigma_{2,3})\wedge(\mu_{1,2}\neq\mu_{1,3}\neq\mu_{2,3})$$

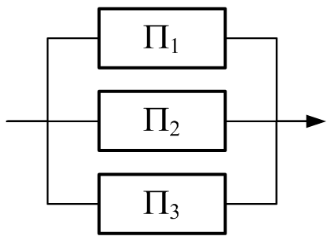

Figure 5. Subnetwork with three parallel flows of PDM structures with similar characteristics

The complete table of the relational operators for the three parallel flows with all the relational combinations of the expected values μ_ν $(\nu=1,2,3)$ and adequate standard deviations σ_ν is given in Table 2.

The number of combinations of the superponed flows u for the greater number of the elementary flows w, is of the exponential characters and is:

$$u=3^{w+1}\ \{w\geq2,\ w\in N\}. \tag{12}$$

Table 2.Combination of relations of the three expected values and three standard deviations

$\mu_1 \rho \mu_2 \rho \mu_3$	$\sigma_1 \rho \sigma_2 \rho \sigma_3$	$\mu_1 \rho \mu_2 \rho \mu_3$	$\sigma_1 \rho \sigma_2 \rho \sigma_3$	$\mu_1 \rho \mu_2 \rho \mu_3$	$\sigma_1 \rho \sigma_2 \rho \sigma_3$
< <	< <	= <	< <	> <	< <
< <	< =	= <	< =	> <	< =
< <	< >	= <	< >	> <	< >
< <	= <	= <	= <	> <	= <
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- The equivalence of the w - parallel flows is realized when the following conditions are fulfilled (Figure 5):

$$\left[\forall \mu_{\nu} = \mu_{\nu+1}, \nu = \overline{1, w-1}\right] \wedge \left[\forall \sigma_{\nu} = \sigma_{\nu+1}\right]$$

$$\Rightarrow (\mu_{1,2\ldots w} \equiv \mu_{2,1\ldots w} \equiv \ldots \equiv \mu_{w,w-1\ldots 1}) \wedge \qquad (13)$$

$$(\sigma_{1,2\ldots w} \equiv \sigma_{2,1\ldots w} \equiv \ldots \equiv \sigma_{w,w-1\ldots 1})$$

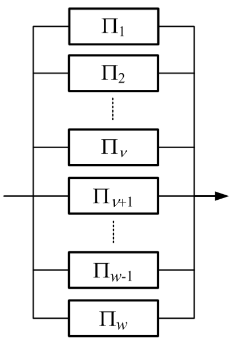

Figure 6. The subnetwork with w - parallel flows

In that sense one can observe these flows as one equivalent flow with the characteristic:

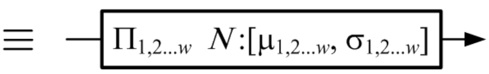

Figure 7. The subnetwork with one equivalent flow

According to the previous criteria, only one case fulfills the condition of equivalence which is given in bold in the table 2. In the paper [5] it is shown how the equivalent superponed flows integrated with the ordinal flows are formed.

6. CONCLUSION

The most significant advantage of Monte-Carlo simulation method in solving this flow problem through the network is the possibility of modelling of the probability distribution function for the superponed flow time of the basic network model, given in figure 1. However, the advantage of Monte-Carlo simulation method is substantially increased on the account of possible dynamic modeling of the flow through the network. The frames made by scanning in the Mathcad provide more reliable basis for further acquiring and expanding of knowledge in this field, especially in relation to the relativity of the critical flow. We can perform the time planning of the critical flows with more credibility when using the combined procedures: analytical Clark's equations and numerical Monte-Carlo simulation than when achieving it by the standard procedures of network planning and managing, e.g. through PDM (Precedence Diagramming Method). With the classical PDM, the flow time planning is established on the expected values of the elementary flow times, which leads to a considerable mistake in planning, since the influence of the subcritical flows on forming the total superponed flow time is, in principle, neglected and super- positioned extracts $\Delta\mu$ and $\Delta\sigma$ are reduced to zero. It can be proved that, in the flow network with ten critical flows of the autonomous type, the resulting flow time increases by 11% from the time one should get when calculating by the PDM method. This "mistake in planning", as a theoretical result also verified by simulation for the two parallel flows, is $\delta = 8{,}92\,\%$. The result obtained in this way is not uniform, but it depends on the chosen value pairs of the numerous $\Pi_{\nu} : [\mu_{\nu}, \sigma_{\nu}] \quad (\nu = 1, 2, \ldots w)$. Of course, it is possible to examine the remaining cases, too, through analytical and / or numerical methods, e.g. when we set the vector of the expected values and of the corresponding standard deviations in the next effect:

$$\mu_{\nu} \begin{Bmatrix} < \\ = \\ < \end{Bmatrix} \mu_{\nu+1} \text{ and } \sigma_{\nu} \begin{Bmatrix} < \\ = \\ < \end{Bmatrix} \sigma_{\nu+1} \text{ for } \nu = \overline{1, w-1} \qquad (14)$$

These influences (Figure 6 and 7) can exceptionally be noticed [2], [8] by simulation at more complex ADM (PDM) networks. When the more complex flows are calculated (Figure 8), analytically or by simulation and respecting the developed criteria, one gets interesting values, because here the networks with both the ordinal and parallel flows are comprised. Example, for the initial information:

$$\mu_{\nu} = 100 \text{ and } \sigma_{\nu} = 10 \text{ for } \nu = 1, 2, \ldots, 7 .$$

- The analytical results are:

$$N[\mu = 413{,}4086;\ \sigma = 14{,}16232],$$

- The numerical results are:

$$N[m = 413{,}3794;\ s = 14{,}17712].$$

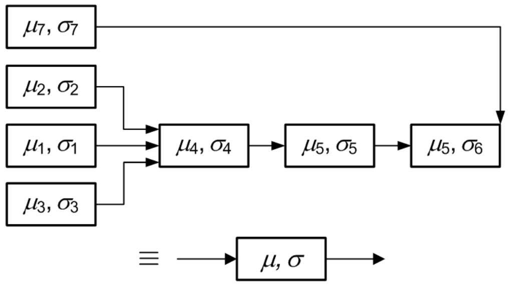

Figure 8. The network with ordinally parallel flows and its equivalent

The consequences of not knowing the essence of the obtained results can be very problematic, especially in the cases of planning and controlling the complex stochastic flows of activities through the network. Clark's equations for four and more parallel flows were not developed. If they were performed, they would lead to the very complex equations. However, the existing equations for two or three parallel flows can be used for solving even the more complex cases of flows ($w \geq 3$). Then they are used as recurrent. *For* w^{th} iteretion, the superposing of the flow $\Pi_{12\ldots w-1}$ and Π_w into the flow $\Pi_{12\ldots w-1,w}$ gives the following results:

- The expected superponed flow time $\overline{T_{12\ldots w-1,w}}$ is:

$$\overline{T_{12\ldots w-1,w}} = \overline{T_{12\ldots w-1}} \cdot \Phi(\xi_{12\ldots w-1,w}) + \overline{T_w} \cdot \Phi(-\xi_{12\ldots w-1,w}) + \lambda_{12\ldots w-1,w} \cdot \varphi(\xi_{12\ldots w-1,w}). \quad (15)$$

- The superponed dispersion $\sigma^2_{12\ldots w-1,w}$ is:

$$\sigma^2_{12\ldots w-1,w} = (\overline{T_{12\ldots w-1}}^2 + \sigma^2_{12\ldots w-1}) \cdot \Phi(\xi_{12\ldots w-1,w}) + (\overline{T_w}^2 + \sigma^2_w) \cdot \Phi(-\xi_{12\ldots w-1,w}) + (\overline{T_{12\ldots w-1}} + \overline{P_w}) \cdot \cdot \lambda_{12\ldots w-1,w} \cdot \Phi(\xi_{12\ldots w-1,w}) - \overline{T_{12\ldots w-1,w}}^2$$

Note: The underlined values are the expected or the average values of the arguments.

Because of the presences as well as of the ordinal flows, the mathematical and simulation models should be completed with the results of the *central limit theorem* [7].

More importance of the research results are:

a) Provided a new approach to studying the impact of parallel flows in the resulting production flow.
b) Introduced a superponed flow, and defines his time and variance, which are the basic parameters for sizing the intralogistic's capacity.
c) Very credible be determined the time of the schedule implementation during process.
d) Created the basis for measuring risk that a particular course will not be realized within the planned time.
e) The significance of the relativity of the critical and subcritical flow.

The directions of further researches can be added.

a) The solving of the Clark's more complex model when two parallel activities are correlated.
b) The development of Clark's analytical model with three or more parallel flows into one equivalently-superponed flow.

REFERENCES

[1] Clark, C. E., 1961. *The Greatest of Finite Set of Random Variables*, Operation Research, Vol. 9, No. 9, pp: 145-162.

[2] Slyke Van, M. R., 1963. *Monte Carlo Methods and PERT Problem*, Operation Research, Vol. 11, No.5, pp: 839-860.

[3] Dodin, B., 1984. *Determining the (k) Most Critical Paths in PERT Networks*, Operation Research, Vol. 32, No. 4, pp: 859-877.

[4] Haga, A. W., O'keefe, T., 2001.*Crashing PERT networks: A simulation Approach*, 4th International conference of the Academy of Business and Administrative Sciences Conference Quebec City, Canada-

[5] Letić, D., 1996. *Educative and general model of critical flows of material of Precedence Diagramming structure,*(PhD thesis), Technical Faculty "Mihajlo Pupin", Zrenjanin.

[6] Letić, D., Davidović, B., Berković, I., Petrov, T., 2007. *MATHCAD 13 in mathematics and visualisation,* Computer Library, Čačak.

[7] Fishman, S. G., 1986. *A Monte-Carlo Sampling Plan for Estimating Network Reliability*, Operation Research, Vol. 34, No. 4, pp: 581-594.

[8] Letić, D., Davidović, B. and Živković, Z. D., 2013. Determining the Realization Risk of Network Structured Material Flows in Machine Building Industry Production Proces, International Journal of Engineering & Technology IJET-IJENS, Vol: 13, No. 02, pp: 90-93.